Infinite λ-calculus

Chris Purdy & Reuben Rowe

Infinite λ-terms

"The same, but infinite"

λ-calculus & reduction

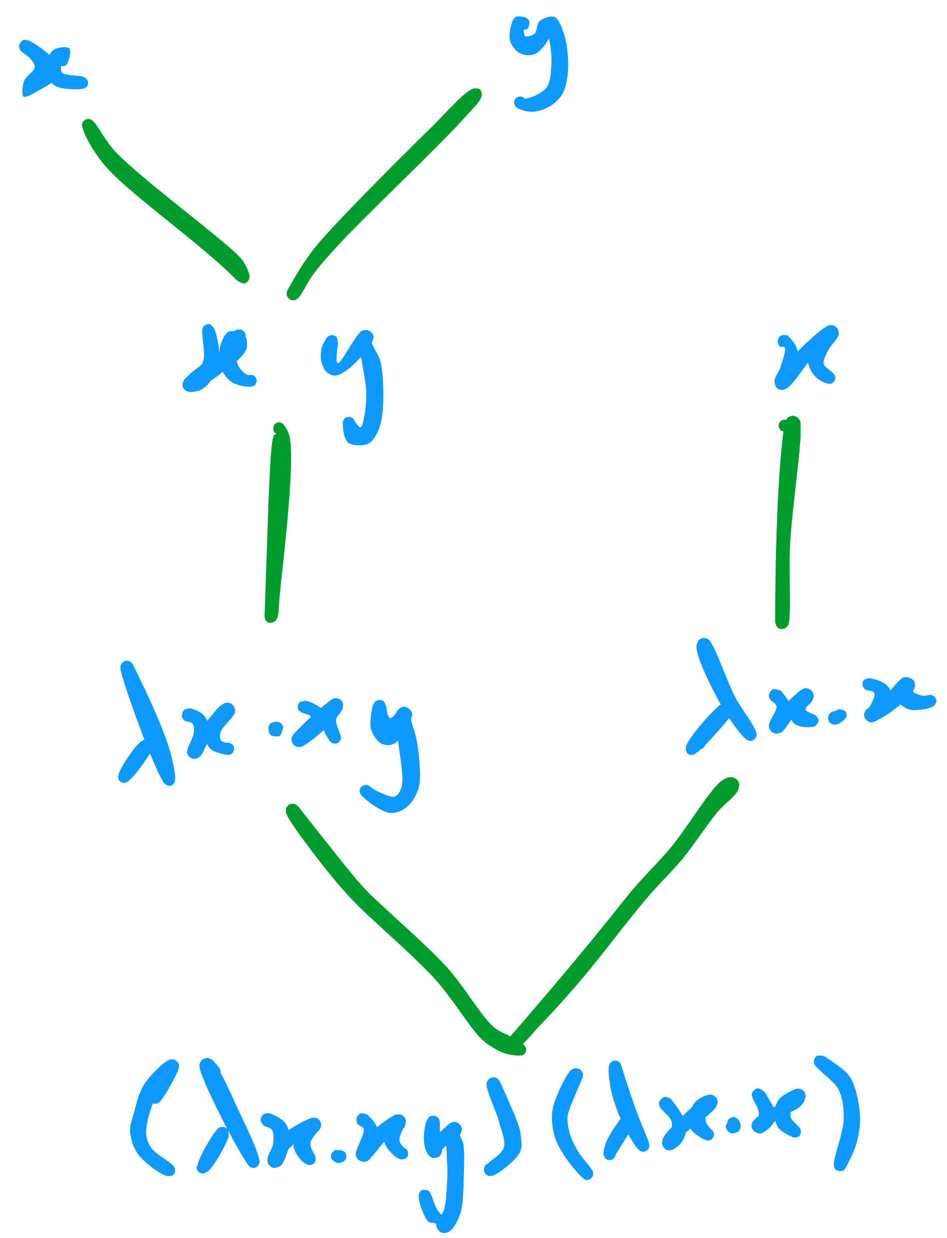
$$(\lambda x.t)\,t' \longrightarrow t[t'/x]$$

$$(\lambda x \lambda y.y)\,a\,b \longrightarrow (\lambda y.y)\,b \longrightarrow b$$

"finite reduction"

Terms are trees

$$(\lambda x. x\, y)\,(\lambda x. x)$$



The language

$$t ::= x \mid t\,t \mid \lambda x.t$$

The usual grammar for (untyped) λ-terms

The language

$$t ::= x \mid t\ t \mid \lambda x.t$$
$$\mid 0 \mid s\,t \mid \text{case } t \text{ of } 0 \Rightarrow t ;\ s x \Rightarrow t$$

3 is represented by $s(s(s\,0))$

n "is" $s^n 0$

[Kennaway et al. 1997]

The language

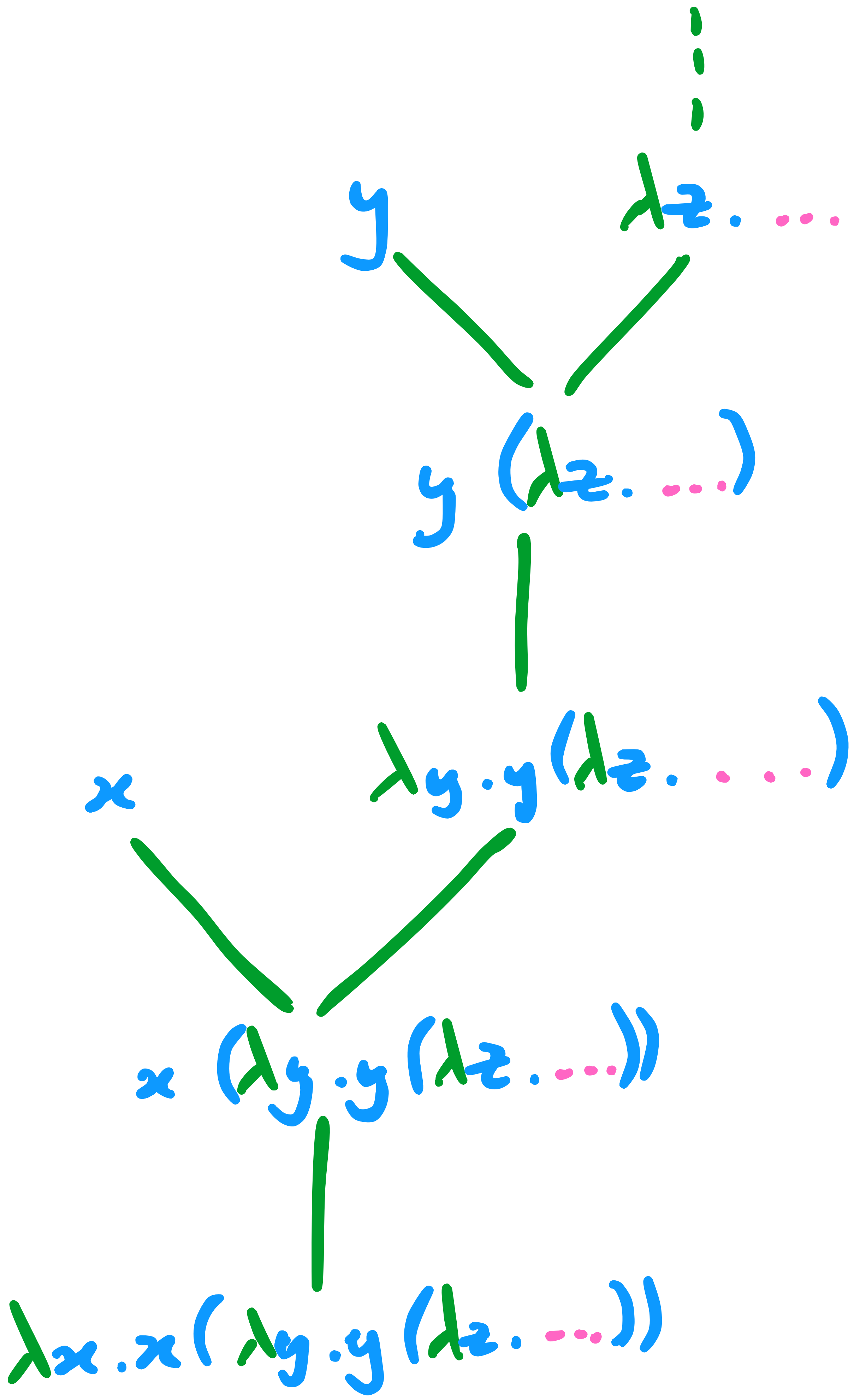
$$t ::= x \mid t\;t \mid \lambda x.t$$
$$\mid 0 \mid s\,t \mid \text{case } t \text{ of } 0 \Rightarrow t ; \; s\,x \Rightarrow t$$

We consider all possibly infinite coterms formed with this grammar, call this $\Lambda^\infty(N)$

[Kennaway et al. 1997]

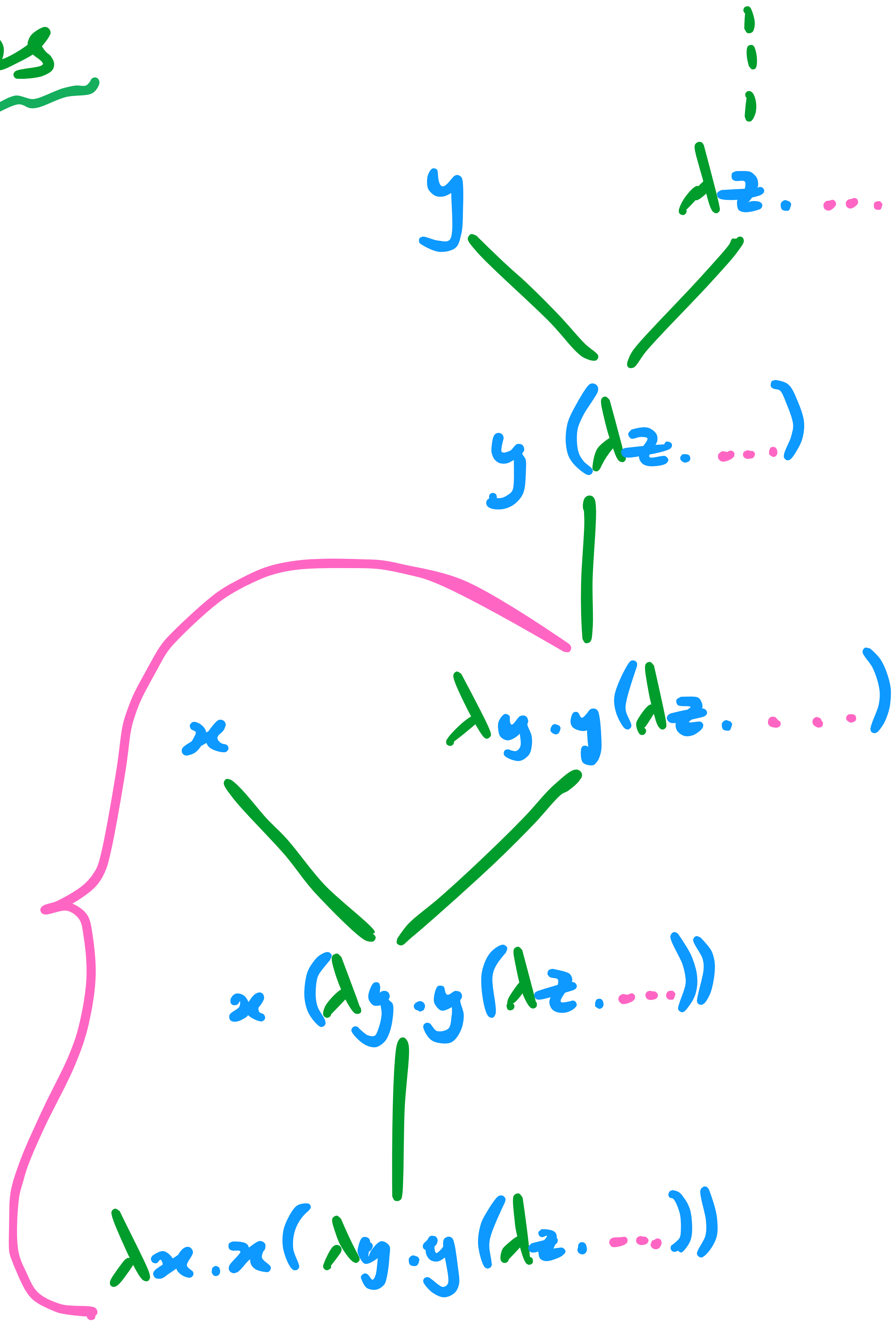
Coterms are (possibly) infinite trees

$$\lambda x.\, x\,(\lambda y.\, y\,(\lambda z. \ldots))$$



Coterms are (possibly) infinite trees

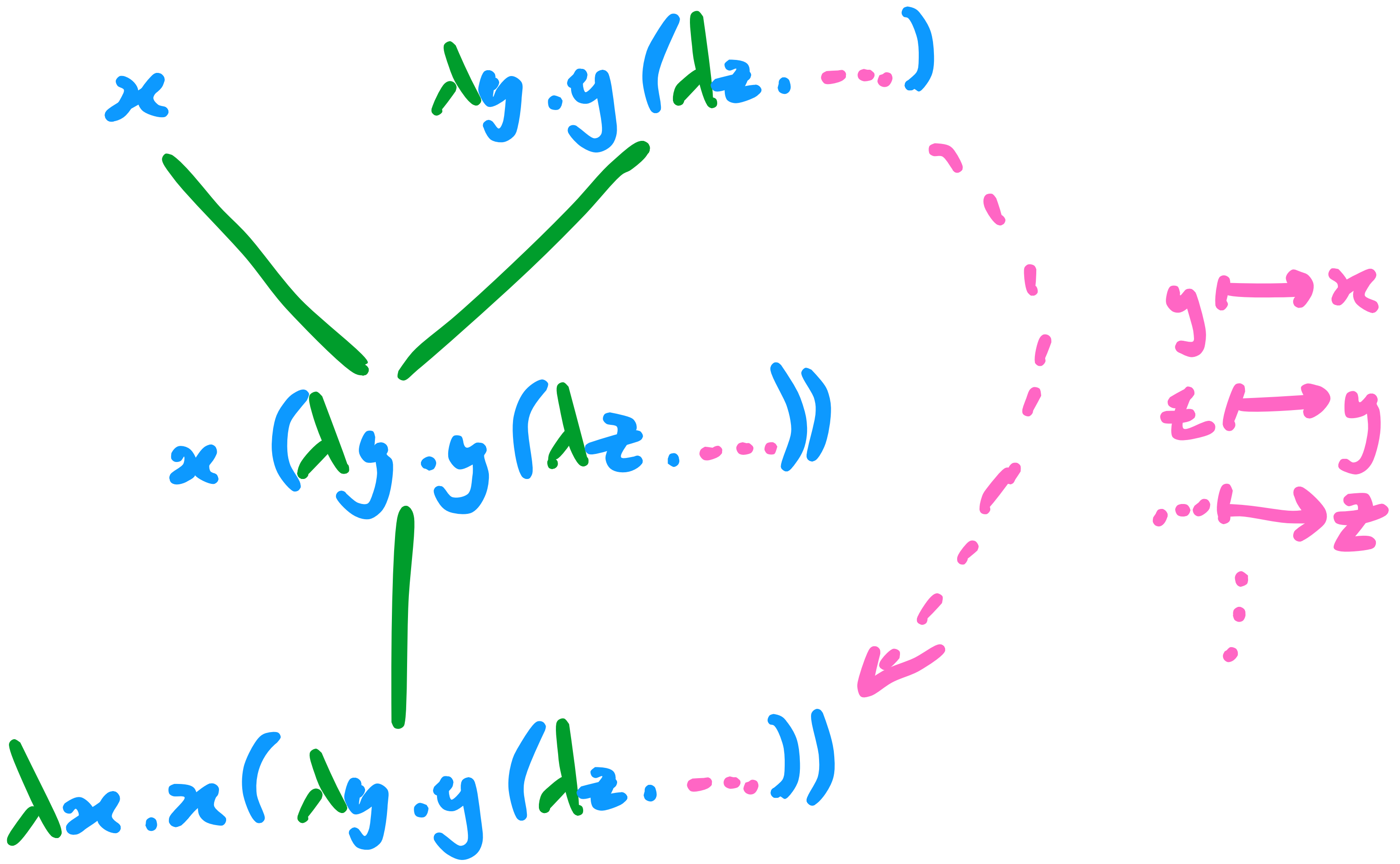
This coterm is "regular", so can be finitely presented.



Repeats, up to renaming of variables.

Coterms are (possibly) infinite trees

This coterm is "regular",
so can be finitely presented.



Coterms are (possibly) infinite trees

$$\lfloor t \rfloor_1 \;, \quad \lfloor t \rfloor_2 \;, \quad \lfloor t \rfloor_3 \;, \ldots$$

$$\lfloor t \rfloor_1 = \lambda x.\, \square \qquad \lfloor t \rfloor_2 = \lambda x. x\, \square \qquad \lfloor t \rfloor_3 = \lambda x. x\, (\lambda y. \square)$$

Lemma

$$t = \lim_{n \to \infty} \lfloor t \rfloor_n$$

Infinite reductions

"It's just an iCRS!"

Infinite reductions

$\Lambda^\infty(N)$ with the following reduction rules:

$$(\lambda x.t)\,t' \longrightarrow t[t'/x]$$

$$\text{case } 0 \text{ of } \begin{array}{l} 0 \Rightarrow t; \\ sx \Rightarrow t' \end{array} \longrightarrow t$$

$$\text{case } s\,t \text{ of } \begin{array}{l} 0 \Rightarrow t'; \\ sx \Rightarrow t'' \end{array} \longrightarrow t''[t/x]$$

... is an iCRS

[Ketema & Simonsen 2009]

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Infinite reductions

"Strongly convergent infinite reductions"

$I := \lambda x . x$

$I^\omega := (\lambda x . x)\, I^\omega$

I^ω can be reduced infinitely many times,
but no progress is made towards a "normal form"

[Kennaway et al. 1997]
[Ketema & Simonsen 2009]

Infinite reductions

"Strongly convergent infinite reductions"

$$\mathrm{add} := \lambda y \lambda x.\ \mathrm{case}\ x\ \mathrm{of}\ \begin{array}{l} 0 \Rightarrow y; \\ S\,x \Rightarrow S\,(\mathrm{add}\ y\ x) \end{array}$$

$$\mathrm{add}\ 3 \longrightarrow \lambda x.\ \mathrm{case}\ x\ \mathrm{of}\ \begin{array}{l} 0 \Rightarrow 3 \\ S\,x \Rightarrow S\,(\mathrm{add}\ 3\ x) \end{array}$$

$$\longrightarrow \lambda x.\ \mathrm{case}\ x\ \mathrm{of}\ \begin{array}{l} 0 \Rightarrow 3 \\ S\,x \Rightarrow S\,(\mathrm{case}\ x\ \mathrm{of}\ \begin{array}{l} 0 \Rightarrow 3 \\ S\,x \Rightarrow S\,(\mathrm{add}\ 3\ x)) \end{array} \end{array}$$

$$\longrightarrow \ldots$$

[Kennaway et al. 1997]
[Ketema & Simonsen 2009]

Infinite reductions

"Strongly convergent infinite reductions"

$$\mathrm{add} := \lambda y \lambda x.\,\mathrm{case}\ x\ \mathrm{of}\ 0 \Rightarrow y;\ \ s\,x \Rightarrow s\,(\mathrm{add}\ y\ x)$$

$$\mathrm{add}\ 3 \xrightarrow{0} \lambda x.\,\mathrm{case}\ x\ \mathrm{of}\ 0 \Rightarrow 3;\ \ s\,x \Rightarrow s\,(\mathrm{add}\ 3\ x)$$

$$\xrightarrow{3} \lambda x.\,\mathrm{case}\ x\ \mathrm{of}\ 0 \Rightarrow 3;\ \ s\,x \Rightarrow s\,(\mathrm{case}\ x\ \mathrm{of}\ 0 \Rightarrow 3;\ \ s\,x \Rightarrow s\,(\mathrm{add}\ 3\ x))$$

$$\xrightarrow{5} \ldots$$

The reduction sequence converges.

[Kennaway et al. 1997]
[Ketema & Simonsen 2009]

Types

"Which coterms compute?"

$$T ::= \mathbb{N} \mid T \to T$$

Grammar of types

$$\frac{}{\Gamma, x : A \vdash x : A}\ (\mathrm{ax})$$

Typing rules

[Barendregt 1992]

Types

"The type N of natural numbers"

$$\frac{}{\Gamma \vdash 0 : N}\ (\text{zero}) \qquad \frac{\Gamma \vdash t : N}{\Gamma \vdash s\,t : N}\ (\text{succ})$$

[Barendregt 1992]

Types

"Type of functions $A \to B$"

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x. t : A \to B} \ (\text{abs})$$

$$\frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash t' : A}{\Gamma \vdash t\, t' : B} \ (\text{app})$$

[Barendregt 1992]

A progressing coderivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{y:N,\, x:N \vdash x:N}(\text{ax}) \quad \dfrac{}{y:N,\, x:N \vdash y:N}(\text{ax}) \quad \dfrac{\dfrac{\vdots}{y:N,\, x:N \vdash \cdots y\, x : N}}{y:N,\, x:N \vdash s(\cdots y\, x):N}(\text{succ})}{y:N,\, x:N \vdash \text{case } x \text{ of } 0 \Rightarrow y;\ s\,x \Rightarrow s(\cdots y\, x):N}(\text{case})}{y:N \vdash \lambda x.\, \text{case } x \text{ of } 0 \Rightarrow y;\ s\,x \Rightarrow s(\cdots y\, x):N \to N}(\text{abs})}{\lambda y \lambda x.\, \text{case } x \text{ of } 0 \Rightarrow y;\ s\,x \Rightarrow s(\cdots y\, x):N \to N \to N}(\text{abs})
$$

Typing the addition function $N \to N \to N$

A progressing coderivation

Progression point

$$\vdots$$

$$y:\mathbb{N}, x:\mathbb{N} \vdash \cdots y\,x : \mathbb{N}$$

$$\dfrac{y:\mathbb{N}, x:\mathbb{N} \vdash x:\mathbb{N}\ (\text{ax}) \qquad y:\mathbb{N}, x:\mathbb{N} \vdash y:\mathbb{N}\ (\text{ax}) \qquad \dfrac{y:\mathbb{N}, x:\mathbb{N} \vdash \cdots y\,x : \mathbb{N}}{y:\mathbb{N}, x:\mathbb{N} \vdash s(\cdots y\,x):\mathbb{N}}\ (\text{succ})}{y:\mathbb{N}, x:\mathbb{N} \vdash \text{case } x \text{ of } 0 \Rightarrow y \,;\, s\,x \Rightarrow s(\cdots y\,x):\mathbb{N}}\ (\text{case})$$

$$\dfrac{y:\mathbb{N}, x:\mathbb{N} \vdash \text{case } x \text{ of } 0 \Rightarrow y \,;\, s\,x \Rightarrow s(\cdots y\,x):\mathbb{N}}{y:\mathbb{N} \vdash \lambda x.\,\text{case } x \text{ of } 0 \Rightarrow y \,;\, s\,x \Rightarrow s(\cdots y\,x):\mathbb{N}\to\mathbb{N}}\ (\text{abs})$$

$$\dfrac{y:\mathbb{N} \vdash \lambda x.\,\text{case } x \text{ of } 0 \Rightarrow y \,;\, s\,x \Rightarrow s(\cdots y\,x):\mathbb{N}\to\mathbb{N}}{\lambda y \lambda x.\,\text{case } x \text{ of } 0 \Rightarrow y \,;\, s\,x \Rightarrow s(\cdots y\,x):\mathbb{N}\to\mathbb{N}\to\mathbb{N}}\ (\text{abs})$$

A (n infinite) coderivation must progress infinitely often.

Properties of coderivations

Uniqueness of typing

If $\Gamma \vdash t : A$ and $\Gamma \vdash t : B$ then $A = B$

Subject reduction

If $\Gamma \vdash t : A$ and $t \longrightarrow t'$ then $\Gamma \vdash t' : A$

"reduction preserves types"

Normalisation/canonicity

Well-typed terms should reduce to a normal form

In particular, $\vdash t : \mathbb{N}$ should mean that $t \longrightarrow s^n 0$ for some natural number n.

Semantics of coderivations (typed cuterms)

$$[\![N]\!]^{R_L} = \mathbb{N} \qquad [\![A \to B]\!]^{R_L} = \overbrace{[\![A]\!]^{R_L} \to [\![B]\!]^{R_L}}^{\text{Set of functionals}}$$

Let $\delta = \langle \vdash t : A \rangle$.

$[\![\delta]\!]^{R_L}$ is (initially) a partial functional in the set $[\![A]\!]^{R_L}$

Semantics of coderivations (typed coterms)

Theorem

For a progressing coderivation $\delta = \langle \vdash t : A \rangle$ we have that $[\![\delta]\!]^{\mathbb{R}_1}$ is total.

Corollary

Given a progressing coderivation $\delta = \langle \vdash t : N \rangle$, there exists some $n \in \mathbb{N}$ where $[\![\delta]\!]^{\mathbb{R}_1} = n$

Relation to Gödel's System T

System T [Tait] $\longleftrightarrow$ Cyclic T [Das]

$$K\,x\,y \longrightarrow x$$

$$S\,x\,y\,z \longrightarrow x\,z\,(y\,z)$$

$$R\,x\,y\,0 \longrightarrow x$$

$$R\,x\,y\,(\mathrm{Succ}\,z) \longrightarrow y\,z\,(R\,x\,y\,z)$$

[Das 2021]

Relation to Gödel's System T

System T [Tait]

$K\,x\,y \to x$

$S\,x\,y\,z \to x\,z\,(y\,z)$

$R\,x\,y\,0 \to x$

$R\,x\,y\,(\mathrm{Succ}\,z) \to y\,z\,(R\,x\,y\,z)$

Cyclic T [Das]

Kept the same*

$\mathrm{cond}\,x\,y\,0 \to x$

$\mathrm{cond}\,x\,y\,(\mathrm{Succ}\,z) \to y\,z$

+ regular infinite (co)terms!

[Das 2021]

Relation to Gödel's System T

Cyclic T [Das] $\longleftrightarrow$ Cyclic Natural Deduction

$K\,x\,y \to x$

$S\,x\,y\,z \to x\,z\,(y\,z)$

$\mathrm{cond}\ x\ y\ 0 \to x$

$\mathrm{cond}\ x\ y\ (\mathrm{Succ}\ z) \to y\,z$

[Das 2021]

Relation to Gödel's System T

Cyclic T [Das]

$K x y \to x$

$S x y z \to x z (y z)$

$\mathrm{cond}\ x\ y\ 0 \to x$

$\mathrm{cond}\ x\ y\ (\mathrm{Succ}\ z) \to y z$

Cyclic Natural Deduction

$(\lambda a \lambda b . a)\, x y \to x$

$(\lambda a \lambda b \lambda c . a c (b c))\, x y z \to x z (y z)$

$\mathrm{case}\ 0\ \mathrm{of}\ 0 \Rightarrow x;\ s z \Rightarrow y z \longrightarrow x$

$\mathrm{case}\ s z\ \mathrm{of}\ 0 \Rightarrow x;\ s a \Rightarrow y a \longrightarrow y z$

[Das 2021]

Parallel work

[Berardi 2026] discusses normalisation for a very similar system.

Further directions

Extend to a dependantly typed system with inductive/coinductive predicates.

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Extend to a dependantly typed system with inductive/coinductive predicates.

Thank you! :)